Вопросы тэории электрических цепей с переменными параметрами и синтеза импульсных и цифровых автоматических регуляторов

В. А. ТАФТ

ELECTRICAL CIRCUITS WITH VARIABLE PARAMETERS

including

PULSED-CONTROL SYSTEMS

by

V. A. TAFT

Translated by

F. IMMIRZI

Translation edited by

R. C. GLASS

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PREFACE

THE results outlined in this work enable us to extend the frequency-analysis and operational methods to circuits with variable parameters. In addition, these results lend themselves in a number of cases to further development in connection with non-linear circuits.

As the work is intended primarily for engineers, several mathematical details have been omitted. The reader can find these details in other publications.

The author will be grateful for all observations. These are to be addressed to the Natsional'nyi Komitet I.F.A.C., Kalanchevskaya ulitsa, Moscow.

V. A. TAFT

INTRODUCTION

THE theory of circuits with variable parameters is of great value in a number of important engineering problems, both directly and in the study of periodic modes of operation of non-linear systems.

Problems involving the consideration of circuits with variable parameters occur, in particular, in electrical and radio engineering in the study of parametric oscillators, in the investigation of processes occurring in synchronous machines, in the investigation of parametric oscillations in non-linear circuits fed by a sinusoidal voltage, in the study of oscillation generators and also in the solution of the problem of frequency stabilization, in the design of parametric amplifiers and trigger circuits etc.

The theory of circuits with variable parameters is of great importance in the theory of automatic control. Not only are a number of important devices to be controlled, themselves systems with variable parameters, but pulse and digital control systems are also essentially systems with variable parameters. The latter systems are finding wider and wider application today, owing to the rapid development of digital techniques and the great possibilities opened up by their use.

We should mention that, in spite of the fact that the analysis of pulsed-control systems for the general case (when the equipment to be controlled has variable parameters) as well as the analysis of periodic modes of operation are both very important and urgent problems, considerable gaps exist in the theory at present.

In particular, in the investigation of periodic modes of operation the problem[3] of whether the frequency-analysis approach may be used for the study of their stability still remains to be clarified.

The investigation of the stability of periodic modes of operation reduces to the investigation of the stability of the solutions of differential equations with periodic coefficients. Much attention has been devoted to this problem, but no rigorous solution for the general case is known. Attempts have been made comparatively recently to use, for the solution of this problem, certain graphical methods based on the frequency-analysis approach.[1,2]

In particular, the problem, formulated by M.A. Aizerman,[3] of obtaining an approximate solution, based on the assumption of the presence in the system investigated of a linear element having the properties of an ideal filter, has been considered previously.[4] Such an approximate solution, however, does not enable one to obtain directly an answer to the problem of calculating the actual, non-idealized, frequency characteristic of the linear part of the system.

A rigorous solution of the problem is given in this work without any assumption as to the presence of an ideal filter.[6]

The solution of the problem reduces to giving to the characteristic equation of a system of equations with periodically varying coefficients, written down initially in the form of an infinite determinant, a finite form, by using a method analogous to the one used in deriving Hill's equation.

Firstly a system is considered with many degrees of freedom with one non-linear parameter and a method is given for reducing the characteristic equation for the equations of the small deviations from a periodic motion to a finite form. After the characteristic equation has been reduced to a finite form, known frequency-analysis criteria can be used for the analysis of stability. As is shown in this monograph, the method described can be extended to the case of several equations with periodic coefficients. The results obtained can be used in the solution of the engineering problems mentioned above.

A very convenient method in the theory of pulsed and digital control, enabling one to analyse a control system rapidly and clearly or to synthetize it, i.e. to determine the pulse characteristic of the transfer coefficient or alternatively the program in a digital controller, is the z-transformation method. In the case, however, when the system to be controlled is a system with variable parameters, the use of the z-transformation methods involves considerable difficulties. These difficulties are connected with the fact that, in order to determine the z-transform of a circuit with variable parameters, it is necessary to know the circuit response to an applied disturbance, or else the L-transform of this response. In order to determine the response by the usual methods, it is necessary to solve a differential equation with variable coefficients.[15] In the case of systems with variable parameters there are considerable difficulties also in determining the L-transform, the difficulties being connected with the use of operational methods in the solution of this problem, since in these circuits, in contrast to the case of circuits with constant parameters, one cannot isolate the system function in a closed form. For example, instead of the usual relation for a quadrupole with constant parameters $U_2(p) = U_1(p)K(p)$, more complex relations occur of the form $U_2(p) = F[U_1(p)\cdot \cdot K_i(p)]$ where $K_i(p)$ are the transfer coefficients of individual elements of the system and F is a functional dependence which in the general case can be very complicated.

In order to introduce a function equivalent, to some extent, to the system function of a circuit with constant parameters, the transient response to a unit impulse function is considered.

Bearing in mind that the transform of the δ-function is equal to unity, in the expression for the response to a δ-function input there only occur the values of the parameters of the system. The determination of this response, however, also gives rise in the general case, to considerable analytical difficulties.[19]

In particular, the L-transform of this response for systems with periodically or exponentially varying parameters can be expressed by means of infinite determinants. Complicated expressions of this type do not enable one to pass directly to the z-transforms.

A method is indicated in this work which enables one to pass from the expression of the L-transform of the response indicated, in the form of the ratio of two infinite determinants, to finite transcendental functions of the operator p, which enables one to reduce the theory of circuits with variable parameters to a form, similar to the theory of transmission lines. The relations obtained enable one to pass to the z-transforms, and the extension of the z-transformation methods to systems with variable parameters becomes possible.

In addition to the indicated extension of z-transformation theory, the analysis of pulsed and digital systems on the basis of the theory of circuits with variable parameters affords a deeper understanding of the properties of pulse and digital automatic-control systems, which in the general case are by their very nature systems with variable parameters.

Alongside with the extension of z-transformation theory, based on the direct evaluation of the L-transform of the response of a system to a δ-function input, we can carry out this extension by using the method of Fourier series (the method of reduction to steady-state modes of operation) in those cases when what matters is the behaviour of the system during a finite interval of time.[11, 12, 5] A sufficiently effective general method of solution of the problem can be obtained for the case of a finite interval.

The monograph comprises six chapters.

The first chapter considers the response of a circuit with periodically varying parameters (namely a periodically varying inductance; the method employed is however, also applicable to the general case when all parameters are periodically varying). Firstly a simple resonant circuit having a sinusoidally varying inductance and fed by a sinusoidal e.m.f. is investigated, and more complex cases are then examined when the variable parameter varies according to an arbitrary periodic law and the applied e.m.f. varies periodically. The latter case corresponds to the experimental method of determining a frequency characteristic.[7] In the same chapter the case of a complex circuit with many periodically varying parameters is also considered.

In the second chapter the free oscillations of a system with periodically varying parameters are investigated. A method is given for reducing the characteristic equation, written in the form of an infinite determinant, to a finite form. It is shown how it is possible to extend the results obtained to systems with monotonically varying parameters by replacing the assigned time-dependence of the parameters approximately by the sum of exponential functions.

The third chapter is devoted to the problem of the application of the operational calculus to circuits with variable parameters. The transform of the response of a circuit with periodically or exponentially varying parameters to a δ-function input is obtained here in a finite form, as well as an expression for the response of such a system to an arbitrary disturbance.

The principles and foundations of the calculation of the transients in circuits with constant and variable parameters by means of the Fourier-series method are treated in the fourth chapter.

The fifth chapter is devoted to the theory of pulsed systems, these being a particular case of systems with periodically varying parameters. First the basic principles of the methods for the analysis and synthesis of these systems on the basis of the z-transformation are outlined,[10, 16, 18, 19] and then, on the basis of the material expounded in the third and fourth chapters, it is shown how the method can be extended to the case of systems with variable parameters.

The sixth chapter is devoted to investigating the stability of circuits with variable parameters. Frequency-analysis methods for the investigation of the stability of automatic-control systems containing variable parameters are given.

CHAPTER I

THE FORCED CURRENT COMPONENT IN AN OSCILLATORY CIRCUIT WITH A PERIODICALLY VARYING INDUCTANCE

LET us consider the forced oscillations in a simple oscillatory circuit with a periodically varying inductance L, a constant resistance r and a capacitance C (Fig. 1).

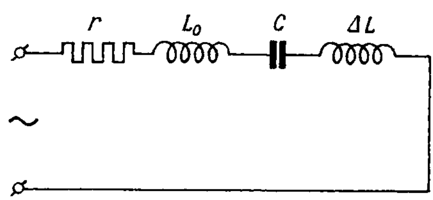

FIG. 1. Simple resonant circuit

If the inductance L were also a constant quantity, then, after connecting to the circuit a source of sinusoidal e.m.f. of frequency ω_0 oscillations would arise in the system of the same frequency. Resonance is possible in this circuit (for $r=0$) only when the frequency ω_0 of the e.m.f. source coincides with the frequency of free oscillations $\omega = 1/\sqrt{(LC)}$.

If the inductance L is not a constant quantity but a periodic function of time, the forced oscillations and the resonance phenomena occurring in such a circuit have a number of typical features. As will be seen below, these features are connected with the fact that, owing to the periodic variation of the parameters, the applied e.m.f., the frequency of which is ω_0, gives rise here to an infinite number of harmonic oscillations with combination frequencies $\omega_0+k\Omega$, where $k = -\infty, \ldots, -1, 0, 1, \ldots, \infty$, each of which can cause the occurence of resonance phenomena.

Let the inductance vary according to the relation

$$L(t) = L_0[1+m\cos(\Omega t+\alpha)], \tag{1.1}$$

where Ω is the frequency of variation of the parameter L, and m is the modulation depth.

Suppose that, in addition, the electromotive force of the source varies according to the relation

$$u(t) = U\cos(\omega_0 t+\theta_0) = \\ = \frac{U}{2}\left[e^{j(\omega_0 t+\theta_0)}+e^{-j(\omega_0 t+\theta_0)}\right], \tag{1.2}$$

where U is the amplitude, ω_0 is the angular frequency and θ_0 is the initial phase.

In this case the forced current component in the circuit (Fig. 1) will be determined as a particular solution of a linear differential equation with periodically varying coefficients

$$\frac{d}{dt}[L(t)\cdot i(t)]+r\cdot i(t)+\frac{1}{C}\int i(t)dt = u(t) \tag{1.3}$$

or

$$L(t)\frac{di(t)}{dt}+\left[r+\frac{dL(t))}{dt}\right]i(t)+\frac{1}{C}\int i(t)dt = u(t). \tag{1.4}$$

In the equations (1.3) and (1.4) $L(t)$ is a function of time defined by the expression (1.1).

We shall seek a particular solution of equation (1.3) in the form of the sum of the terms of a Fourier series

$$i(t) = \sum_{k=-\infty}^{\infty} I_k\cos[(\omega_0+k\Omega)t+\varphi_k]; \tag{1.5}$$

$$i(t) = \frac{1}{2} e^{j\omega_0 t} \sum_{k=-\infty}^{\infty} \dot{I}_k e^{jk\Omega t} + \frac{1}{2} e^{-j\omega_0 t} \sum_{k=-\infty}^{\infty} \dot{I}_{-k} e^{-jk\Omega t}, \tag{1.6}$$

where

$$\dot{I}_k = I_k e^{j\varphi k}; \quad \dot{I}_{-k} = I_k e^{-j\varphi k}. \tag{1.7}$$

We observe that $\dot{I}_k$ and $\dot{I}_{-k}$ are two complex conjugate numbers. In connexion with this we shall consider below the first term only of (1.6) and accordingly shall seek a solution of (1.3) in the form of the real part of the first term of (1.6). Since, according to (1.1),

$$L(t) = L_0\left(1 + m\frac{e^{j(\Omega t+\alpha)} + e^{-j(\Omega t+\alpha)}}{2}\right),$$

then

$$[L(t)\cdot i(t)] = \mathrm{Re} L_0 \left[\sum_{k=-\infty}^{\infty} \dot{I}_k e^{j(\omega_0 + k\Omega)t} + \frac{m}{2}\sum_{k=-\infty}^{\infty} \dot{I}_k e^{j\{[\omega_0+(k+1)\Omega]t+\alpha\}} + \frac{m}{2}\sum_{k=-\infty}^{\infty} \dot{I}_k e^{j\{[\omega_0+(k-1)\Omega]t-\alpha\}}\right]. \tag{1.8}$$

In the second and third sums on replacing the index k by k' and k'' respectively according to the relations

$$k' = k+1; \qquad k'' = k-1; \tag{1.9}$$

we have

$$L(t)\cdot i(t) = \mathrm{Re} L_0 \sum_{k=-\infty}^{\infty}\left(\dot{I}_k + \frac{\dot{m}}{2}\dot{I}_{k-1} + \frac{\overset{*}{m}}{2}\dot{I}_{k+1}\right) e^{j(\omega_0+k\Omega)t}, \tag{1.10}$$

where $\dot{m} = me^{j\alpha}$, $\overset{*}{m} = me^{-j\alpha}$ and $\dot{I}_k$ is the complex amplitude of the current oscillation of frequency $e^{j(\omega_0+k\Omega)t}$. By substituting in (1.3) the expression for the current in the form of the first sum of (1.6) and the expression (1.10) for $L(t)\cdot i(t)$, we obtain

$$\sum_{k=-\infty}^{\infty}\left\{\left[j(\omega_0+k\Omega)L_0+r+\frac{1}{j(\omega_0+k\Omega)C}\right]\dot{I}_k+\right.$$

$$\left.+j(\omega_0+k\Omega)L_0\left(\frac{\dot{m}}{2}\dot{I}_{k-1}+\frac{\overset{*}{m}}{2}\dot{I}_{k+1}\right)\right\}e^{j(\omega_0+k\Omega)t}=Ue^{j(\omega_0t+\theta_0)}. \quad (1.11)$$

Since equation (1.11) must be satisfied for arbitrary values of t, it is resolved into an infinite number of recurrent equations of the following form

$$\frac{m}{2}jL_0\dot{I}_{k-1}+\left(jL_0+\frac{r}{\omega_0+k\Omega}+\frac{1}{j(\omega_0+k\Omega)^2C}\right)\dot{I}_k+$$

$$+\frac{m}{2}jL_0\dot{I}_{k+1}=\frac{\dot{U}(k)}{\omega_0+k\Omega},\ k=(-\infty,\ldots,-1,0,1,\ldots,\infty) \quad (1.12)$$

where

$$\dot{U}(0)=U_m e^{j\theta_0};\ U(k)=0 \text{ for } k\neq 0.$$

Let us consider now a particular case, which is very important in what follows. Let

$$\Omega = n\omega_0, \quad (1.13)$$

where n is an integral number. In this case equation (1.8) takes the following form

$$L(t)\cdot i(t) = \mathrm{Re}L_0\left[\sum_{k=-\infty}^{\infty}\dot{I}_k\cdot e^{j(1+kn)\omega_0t}+\right.$$

$$\left.+\frac{\dot{m}}{2}\sum_{k=-\infty}^{\infty}\dot{I}_k\cdot e^{j[1+(k+1)n]\omega_0t}+\frac{\overset{*}{m}}{2}\sum_{k=-\infty}^{\infty}\dot{I}_k\cdot e^{j[1+(k-1)n]\omega_0t}\right]. \quad (1.14)$$

In equation (1.14) $\dot{I}_k$ in the first sum is the complex amplitude of the oscillation of frequency $(1+kn)\omega_0$, in the second sum is the complex amplitude of the oscillation of frequency $[1+(k+1)n]\omega_0$, and in the third sum is the complex amplitude of the oscillation of frequency $[1+(k-1)]n\omega_0$. Let us introduce a new notation in the indices in the second and third sums of

(1.14) by substituting $k'n$ for $(k+1)n$ in the second sum and $k''n$ for $(k-1)n$ in the third sum. Having introduced this new notation in the indices in the second and third sum we must replace the suffixes of the complex amplitudes as follows: k by $k-1$ in the second sum and k by $k+1$ in the third sum.

We then obtain from (1.14) after the transformations indicated

$$L(t)\cdot i(t) = \mathrm{Re} L_0 \sum_{k=-\infty}^{\infty} \left(\dot{I}_k + \frac{m}{2} \dot{I}_{k-1} + \frac{\overset{*}{m}}{2} I_{k+1} \right) e^{j(kn+1)\omega_0}. \quad (1.15)$$

Let us agree in this case to attribute to the complex amplitudes suffices corresponding to the frequency of the oscillation which has the given complex amplitude.

We thus obtain instead of (1.11)

$$\sum_{k=-\infty}^{\infty} \left\{ \left[j(kn+1)\omega_0 L_0 + r + \frac{1}{j(kn+1)\omega_0 C} \right] \dot{I}_{kn+1} + \right.$$

$$\left. + j(kn+1)\omega_0 L_0 \left(\frac{\dot{m}}{2} \dot{I}_{(k-1)n+1} + \frac{\overset{*}{m}}{2} \dot{I}_{(k+1)n+1} \right) \right\} e^{j(kn+1)\omega_0 t} =$$
$$= U e^{j(\omega_0 t + \theta_0)}. \quad (1.16)$$

The equations (1.16) are an infinite system of recurrent equations, from which the complex amplitudes can be determined.

These equations yield an infinite number of equations of the form

$$\frac{\dot{m}}{2} j\omega_0 L_0 \dot{I}_{(k-1)n+1} + \left(j\omega_0 L_0 + \frac{r}{kn+1} + \frac{1}{(kn+1)^2 \omega_0 C} \right) \dot{I}_{kn+1} +$$

$$+ \frac{\overset{*}{m}}{2} j\omega_0 L_0 \dot{I}_{(k+1)n+1} = \frac{\dot{U}(k)}{kn+1}, \; (k = -\infty, \ldots, -1, 0, 1, \ldots, \infty),$$

where

$$U(0) = \dot{U}; \quad \dot{U}(k) = 0 \text{ for } k \neq 0. \quad (1.17)$$

The equations (1.17) also represent an infinite system of recurrent equations.

Let us rewrite in full the equations (1.12) and (1.17). The system (1.12) will then take the following form

$$\left.\begin{aligned}
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{m}{2}jL_0\dot{I}_{-2}+\left(jL_0+\frac{r}{\omega_0-\Omega}+\frac{1}{j(\omega_0-\Omega)^2C}\right)\dot{I}_{-1}+\\
&\qquad+\frac{\overset{*}{m}}{2}jL_0\dot{I}_0=0;\\
&\frac{m}{2}jL_0\dot{I}_{-1}+\left(jL_0+\frac{r}{\omega_0}+\frac{1}{j\omega_0^2C}\right)\dot{I}_0+\\
&\qquad+\frac{\overset{*}{m}}{2}jL_0\dot{I}_1=\frac{\dot{U}(0)}{\omega_0};\\
&\frac{m}{2}jL_0\dot{I}_0+\left(jL_0+\frac{r}{\omega_0+\Omega}+\frac{1}{j(\omega_0+\Omega)^2C}\right)\dot{I}_1+\\
&\qquad+\frac{\overset{*}{m}}{2}jL_0\dot{I}_2=0,\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}\right\}\quad(1.18)$$

where the suffix 1 of the complex amplitude denotes that this complex amplitude corresponds to an oscillation of frequency $(\omega_0+\Omega)$, the suffix 2 indicates that it corresponds to an oscillation of frequency $(\omega_0+2\Omega)$ etc.

Similarly the system of equations (1.17) written in full will have the form

$$\begin{aligned}
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\frac{\dot{m}}{2}j\omega_0L_0\dot{I}_{1-2n}+\left(j\omega_0L_0+\frac{r}{1-n}+\frac{1}{j(1-n)^2\omega_0C}\right)\dot{I}_{1-n}+\\
&\qquad+\frac{\overset{*}{m}}{2}j\omega_0L_0\dot{I}_1=0;\\
&\frac{\dot{m}}{2}j\omega_0L_0\dot{I}_{1-n}+\left(j\omega_0L_0+r+\frac{1}{j\omega_0C}\right)\dot{I}_1+\\
&\qquad+\frac{\overset{*}{m}}{2}j\omega_0L_0\dot{I}_{1+n}=\dot{U}(0);
\end{aligned}$$

$$\frac{m}{2} j\omega_0 L_0 \dot{I}_1 + \left(j\omega_0 L_0 + \frac{r}{1+n} + \frac{1}{j(1+n)^2\omega_0 C} \right) \dot{I}_{1+n} +$$

$$+ \frac{\overset{*}{m}}{2} j\omega_0 L_0 \dot{I}_{1+2n} = 0, \tag{1.19}$$

. . . .

where the index $1+n$ of the complex amplitude indicates that this complex amplitude corresponds to an oscillation of frequency $(1+n)\omega_0$, the suffix $(1+2n)$ corresponds to the oscillation of frequency $(1+2n)\omega_0$ etc.

It follows from what has been said that a sinusoidal voltage of frequency ω_0 causes in an oscillatory circuit with a sinusoidally varying inductance forced oscillations with combination frequencies

$$\omega_k = | \omega_0 \pm k\Omega |, \tag{1.20}$$

where $k = 0, 1, 2, \ldots, \infty$.

In the particular case when the frequency of the forced oscillations ω_0 and the frequency of variation of the parameter, Ω, are connected with each other by the relation $\Omega = n\omega_0$, forced oscillations will occur with the combination frequencies

$$\omega_k = \omega_0 | 1 \pm kn |, \tag{1.21}$$

where $k = 0, 1, 2, \ldots, \infty$.

It can be shown that in the particular case when $n = 2$ or $n = 1$, the infinite system of equations (1.18) or (1.19) reduces to two independent semi-infinite systems. The independent variables that occur in these systems are complex conjugate quantities.

In fact, if $n = \Omega/\omega_0 = 2$, i.e. $\Omega = 2\omega_0$ the frequency corresponding to the complex amplitude $\dot{I}_{-1}$ in the system of equations (1.18) is equal to $-\omega_0$, i.e. the vectors $\dot{I}_{-1}$ and $\dot{I}_0$ are complex conjugate vectors. Correspondingly also the right-hand sides of those equations of the system in the central term of which there occur respectively the complex amplitude $\dot{I}_0$ and $\overset{*}{I}_{-1}$ are complex conjugate quantities. Rewriting these two equations of the system (1.18) for the particular case considered, we obtain

$$\left.\begin{aligned}
&-\frac{m}{2}j\omega_0 L_0 \overset{*}{I}_1+\left(-j\omega_0 L_0+r+\frac{1}{-j\omega_0 C}\right)\overset{*}{I}_0- \\
&\qquad -\frac{m}{2}j\omega_0 L_0 \dot{I}_0=\frac{\overset{*}{U}m}{2};\\
&\frac{m}{2}j\omega_0 L_0 \overset{*}{I}_0+\left(j\omega_0 L_0+r+\frac{1}{j\omega_0 C}\right)\dot{I}_0+ \\
&\qquad +\frac{m}{2}j\omega_0 L_0 \dot{I}_1=\frac{\dot{U}m}{2}.
\end{aligned}\right\} \quad (1.18a)$$

The first and second equations of the system (1.18a) are complex conjugate. On passing from the complex quantities to their real parts and bearing in mind that the real parts of complex conjugate quantities are equal to each other, it is easily verified that the infinite system of equations (1.18) reduces to two equal systems.

The infinite system of equations can be similarly reduced in the case when $\Omega = \omega_0$. In fact, rewriting the system of equations (1.18), we obtain for this case

$$\left.\begin{aligned}
&-\frac{m}{2}j\omega_0 L_0 \dot{I}_{-3}+\left(-j\omega_0 L_0+r+\frac{1}{-j\omega_0 C}\right)\dot{I}_{-2}- \\
&\qquad -\frac{m}{2}j\omega_0 L_0 \dot{I}_{-1}=\overset{*}{U}_m;\\
&\frac{m}{2}\cdot j0\cdot L_0 \dot{I}_{-2}+\left(j0\cdot L_0+r+\frac{1}{j0\cdot C}\right)\dot{I}_{-1}+ \\
&\qquad +\frac{m}{2}j0 L_0 \dot{I}_0=0;\\
&\frac{m}{2}j\omega_0 L_0 \dot{I}_{-1}+\left(j\omega_0 L_0+r+\frac{1}{j\omega_0 C}\right)\dot{I}_0+ \\
&\qquad +\frac{m}{2}j\omega_0 L_0 \dot{I}_1=\dot{U}_m.
\end{aligned}\right\} \quad (1.18b)$$

As can be seen from the system (1.18b) $\dot{I}_{-1} = 0$ and the remaining two equations are complex conjugate.

Other special cases corresponding to various values of n can be analysed in a similar manner.

It is of interest for what follows to consider the more general case, namely the case of an arbitrary periodic external e.m.f.

Let the applied voltage be equal to

$$u(t) = \operatorname{Re} \sum_{s=1}^{\infty} U_s e^{j(s\omega_0 t + \theta_{0s})}, \tag{1.22}$$

where ω_0 is the angular frequency of the fundamental harmonic of the voltage applied.

As in the preceding case we shall seek, for the s-th harmonic of the applied voltage, a particular solution of the equation (1.3) in the form

$$i_s(t) = \sum_{k=-\infty}^{\infty} I_{sk} \cdot \cos\left[(s\omega_0 + k\Omega)t + \psi_{sk}\right] \tag{1.23}$$

A particular solution of equation (1.3) when all harmonic components are applied will accordingly have the form

$$i(t) = \sum_{s=1}^{\infty} i_s(t) = \sum_{s=1}^{\infty} \sum_{k=-\infty}^{\infty} I_{sk} \cdot \cos\left[(s\omega_0 + k\Omega)t + \psi_{sk}\right]. \tag{1.24}$$

The determination of the amplitudes $\dot{I}_{sk}$ must be carried out here for each of the harmonics of the applied voltage separately. This determination for each of the harmonics is carried out in the same manner as for the case of a sinusoidal applied voltage considered above.

The determination of the complex amplitudes is considerably simplified in the case when the angular frequency and the frequency of variation of the inductance are connected with each other by the relation (1.13). In this case expression (1.24) takes the following form

$$i(t) = \sum_{s=1}^{\infty} \sum_{k=-\infty}^{\infty} I_{s+kn} \cos\left[(s + kn)\omega_0 t + \psi_{sk}\right]. \tag{1.25}$$

We obtain as for (1.15),†

$$L(t)\cdot i(t) = \operatorname{Re} L_0 \sum_{s=1}^{\infty} \sum_{k=-\infty}^{\infty} \left(\dot{I}_{kn+s} + \frac{m}{2} \dot{I}_{(k-1)n+s} + \right.$$

$$\left. + \frac{\overset{*}{m}}{2} \dot{I}_{(k+1)n+s} \right). \qquad (1.26)$$

Here, just as in (1.16), we introduce such a notation for the suffixes of the complex amplitudes that they be equal to the value of the frequency of that oscillation to which the given complex amplitude corresponds.

After substituting (1.25) and (1.26) in (1.3) we obtain

$$\sum_{s=1}^{\infty} \sum_{k=-\infty}^{\infty} \left\{ \left[j(s+kn)\omega_0 L_0 + r + \frac{1}{j(s+kn)\omega_0 C} \right] \dot{I}_{kn+s} + \right.$$

$$\left. + j(s+kn)\omega_0 L_0 \left(\frac{\overset{*}{m}}{2} \dot{I}_{(k+1)n+s} + \dot{I}_{(k-1)n+s} \right) \right\} e^{j(s+kn)\omega_0 t} =$$

$$= \sum_{s=1}^{\infty} \dot{U}_s e^{js\omega_0 t}. \qquad (1.27)$$

The equation (1.27) reduces to n systems of equations of the form

$$\frac{m}{2} j\omega_0 L_0 \dot{I}_{(k-1)n+s} + \left(j\omega_0 L_0 + \frac{r}{kn+s} + \frac{1}{j(kn+s)^2 C} \right) \dot{I}_{kn+s} +$$

$$+ \frac{\overset{*}{m}}{2} j\omega_0 L_0 \dot{I}_{(k+1)n+s} = \frac{\dot{U}_{kn+s}}{kn+s} \qquad (1.28)$$

$$(k = -\infty, \ldots, -1, 0, 1, \ldots, \infty).$$

† The expression (1.25) can be represented, similarly to (1.5) in the form

$$\frac{1}{2} \sum_{s=1}^{\infty} e^{js\omega_0 t} \sum_{k=-\infty}^{\infty}{}' \dot{I}_k e^{jkn\omega_0 t} + \frac{1}{2} \sum_{s=1}^{\infty} e^{-js\omega_0 t} \sum_{k=-\infty}^{\infty} \dot{I}_{-k} e^{-jkn\omega_0 t}. \qquad (1.6a)$$

Here, just as above, only the first term is considered. Accordingly also the sum on the right-hand side will be taken from $s = 1$ to $s = \infty$ only.

The equations (1.28) represent for a given value of s an infinite system of recurrent equations from which the complex amplitudes of all harmonics of the required current caused by the $(s+kn)$-th harmonics of the applied voltage can be determined.

Since the amplitude of the s-th harmonic of current is connected with the amplitudes of only the $(s-n)$-th and $(s+n)$-th harmonics, the system (1.27) reduces, as has been indicated, to n independent infinite systems of equations corresponding to the values

$$s = 1,\ 2,\ 3, \ldots, (n-1),\ n.$$

The first of these systems corresponding to the value $s = 1$ contains the amplitudes

$$\ldots, \dot{I}_{1-2n}, \dot{I}_{1-n}, \dot{I}_1\ \dot{I}_{1+n}, \dot{I}_{1+2n}, \ldots .$$

the second, corresponding to the value $s = 2$, contains

$$\ldots, \dot{I}_{2-2n}, \dot{I}_{2-n}, \dot{I}_2, \dot{I}_{2+n}, \dot{I}_{2+2n}, \ldots;$$

the third, corresponding to the value $s = 3$, contains

$$\ldots, \dot{I}_{3-2n}, \dot{I}_{3-n}, \dot{I}_3, \dot{I}_{3+n}, \dot{I}_{3+2n}, \ldots;$$

and the n-th system contains the amplitudes

$$\ldots, \dot{I}_{-2n}, \dot{I}_{-n}, \dot{I}_0, \dot{I}_n, \dot{I}_{2n}, \ldots .$$

We observe that all pairs of complex amplitudes the suffixes of which differ in their sign are complex conjugate numbers.

The determinants made up of the coefficients of the unknown complex amplitudes, occurring in each of the enumerated independent systems, are different from zero. Therefore all complex amplitudes occuring in the second, third etc. up to and including the n-th system of equations will be identically equal to zero if their right-hand sides are equal to zero.

We observe here that a part of these infinite systems can be excluded from the analysis. In particular the system of equations corresponding to $s = 1$ will contain the complex